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POWER SERIES SOLUTIONS OF ODE

Outline:

1. Introduction
2. Review of Power Series
3. Theory of Power Series Method
4. Existence of Power Series Solutions
5. Power Series Solutions of ODE
6. Operations on Power Series

1. Introduction

The differential equations studied earlier often possess solutions expressible in terms of elementary functions such as polynomials, exponentials and trigonometric. However, many equations arise from physical problem do not have solutions that can be so conveniently expressed.

In this chapter, our goal is to obtain representations for ODE solutions as power series. One of the major benefits of using power series is that it will allow us to solve many ODE involving variable (non-constant) coefficients.

2. Review of Power Series

Power Series

The **power series** (in powers of $x - x_0$) is an infinite series of the form

$$(1)\ldots \quad \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

$a_0, a_1, a_2, \ldots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **center** of the series, and x is a variable.

If in particular $x_0 = 0$, we obtain a power series in powers of x

$$(2)\ldots \quad \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

assuming that all variables and constants are real.

Remembering how Taylor series work will be a very convenient way to get comfortable with power series before we start looking at differential equations.

Taylor Series

If $f(x)$ is an infinitely differential function then Taylor series of $f(x)$ about $x=x_0$ is:

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m$$

Recall that:

$f^{(0)}(x) = f(x)$ & $f^{(m)}(x) = m^{th}\ derivative\ \ of\ \ f(x)$

Example 1:

Taylor series for *f(x)=cos(x)*

According to Taylor series *f(x)=cos(x)* can be expressed:

$$\cos(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m$$

$$= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$= \frac{1}{0!} + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} + \cdots$$

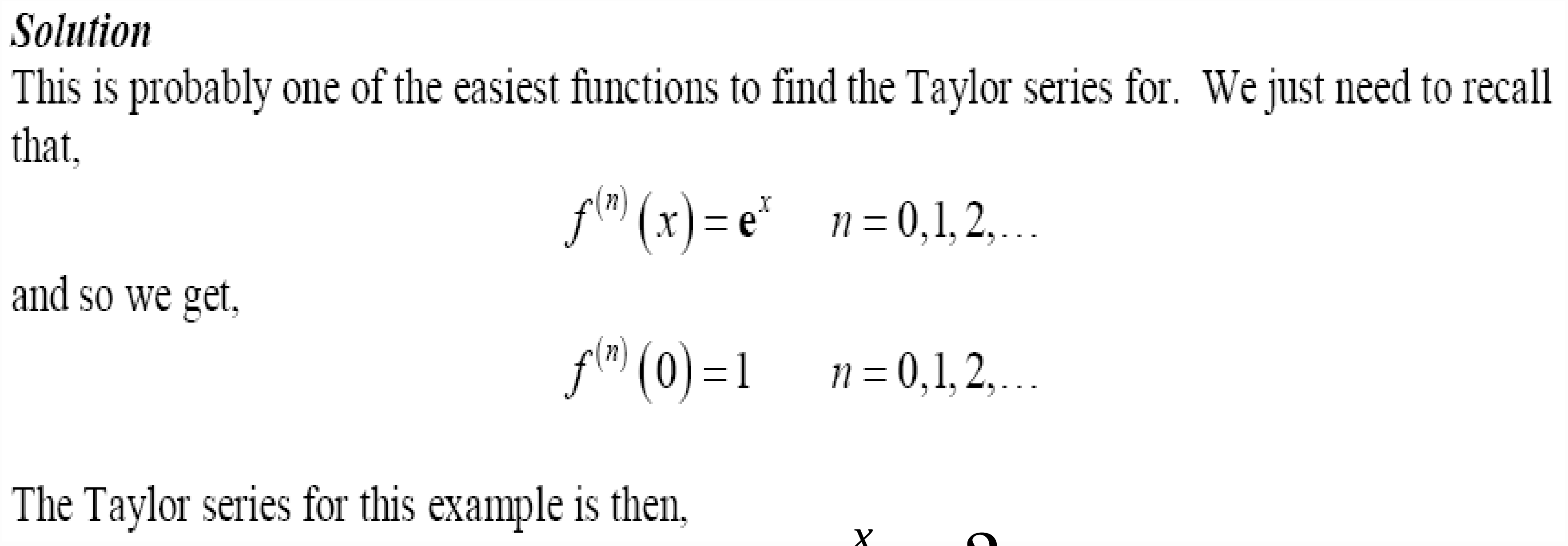
$$\cos(x) = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x^{2m})}{(2m)!}$$

$$\begin{array}{ll}
f^{(0)}(x) = \cos(x) & f^{(0)}(0) = 1 \\
f^{(1)}(x) = -\sin(x) & f^{(1)}(0) = 0 \\
f^{(2)}(x) = -\cos(x) & f^{(2)}(0) = -1 \\
f^{(3)}(x) = \sin(x) & f^{(3)}(0) = 0 \\
f^{(4)}(x) = \cos(x) & f^{(4)}(0) = 1 \\
\vdots & \vdots
\end{array}$$

Example 2:

Determine Taylor series for $f(x)=e^x$ about $x_0=0$.



Solution

This is probably one of the easiest functions to find the Taylor series for. We just need to recall that,

$$f^{(n)}(x)=\mathbf{e}^x \qquad n=0,1,2,\ldots$$

and so we get,

$$f^{(n)}(0)=1 \qquad n=0,1,2,\ldots$$

The Taylor series for this example is then,

$$e^x = ?$$

Familiar examples of power series are MacLaurin series.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \dots \qquad (|x| < 1, \text{ geometric series}),$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

3. Theory of Power Series Method

Convergence and divergence

$$y=\sum_{m=0}^{\infty}a_m(x-x_0)^m=a_0+a_1(x-x_0)+a_2(x-x_0)^2+......$$

If the power series equals a finite real constant for the given *x*, then the series is said to *converge* at *x* (the series is convergent for the given *x*).

If the series does not converge at *x*, it is said to *diverge* at *x* (it is divergent for the given x).

Note that polynomials are also power series.

Convergence Interval & Radius of convergence

Theorem

$$y=\sum_{m=0}^{\infty}a_m(x-x_0)^m=a_0+a_1(x-x_0)+a_2(x-x_0)^2+......$$

be a power series ,then there exists some number 0≤*R*≤∞ ,called its **radius of convergence** such that the series is convergent for → $\left|(x-x_0)\right|<R$

and divergent for → $\left|(x-x_0)\right|>R$

The values of *x*, for which the series converges, form an interval called the **convergence interval.**

Every power series has an interval of convergence.

Radius of convergence

What is R?

The number R is called the **radius of convergence**. It can be obtained as,

$$R = 1 \Big/ \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$$

OR

$$R = 1 \Big/ \lim_{m\to\infty}\sqrt[m]{|a_m|}$$

Radius of convergence

Example 1 The useless Case 1 of convergence only at the center

In the case of the series

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

we have $a_m = m!$ and in $R = 1 \Big/ \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$,

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = m+1 \to \infty \qquad \text{as } m \to \infty.$$

Thus this series converges only at the center $x = 0$. Such a series is useless.

Example 2 Case 2. The geometric series

For the geometric series we have

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x|<1).$$

In fact, $a_m = 1$ for all m, and $R = 1 \Big/ \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|$ we obtain $R = 1$, that is, the geometric series converges and represents $1/(1-x)$ when $|x|<1$.

Example 3 Case 3

In the case of the series

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

we have $a_m = 1/m!$. Hence in $R = 1 \Big/ \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|$

$$\frac{a_{m+1}}{a_m} = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0 \qquad \text{as } m \to \infty,$$

so that the series converges for all x.

Exercise 1:

Determine the radius of convergence, R, for the following power series:

$$\sum_{m=0}^{\infty} \frac{mx^m}{2^{m+1}}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{k^m}$$

4. Existence of Power Series Solutions; Real Analytic Functions

Definition: Ordinary and Singular Points

$$y''+p(x)y'+q(x)y=0 \qquad \text{...(3)}$$

A point x_0 is said to be an ***ordinary point*** of the DE (3) if both *p(x)* and *q(x)* are analytic at x_0. A point that is not an ordinary point is said to be a ***singular point*** of the equation.

What is a real analytic function ?

A real function ***y****(x)* is called ***analytic*** at an ordinary point $x = x_0$ if it can be represented by a power series

$\sum_{m=0}^{\infty} a_m (x-x_0)^m$ with radius of convergence $R > 0$.

Examples

Ordinary and Singular Points

(a) The differential equation $xy'' + (\sin x)y = 0$. A point at $x = 0$ is ordinary point since it can be shown that $q(x) = (\sin x)/x$ possesses the power series form:

$$q(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

that converges for all finite values of x. ($R=\infty$)

(b) The differential equation $y'' + (\ln x)y = 0$ has a singular point at $x = 0$ because $q(x) = \ln x$ possesses no power series in x. (Note: $\ln(0)=-\infty$)

(c) The singular points of the equation $(x^2 - 1)y'' + 2xy' + 6y = 0$ are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. $p(x)= 2x/(x^2 - 1)=\infty$ and also $q(x)= 6/(x^2 - 1)=\infty$ for $x=\pm 1$. All other finite values of x are ordinary points.

(d) $x^2y'' - 2xy' + 2y = 0$ and $x^2y'' - 2y = 0$ has a singular point at $x = 0$ since $p(x)= 2/x=\infty$ and also $q(x)= 2/x^2=\infty$ for $x=0$.

4. Existence of Power Series Solutions; Real Analytic Functions

Existence of the power series solutions

$$y''+p(x)y'+q(x)y=0 \qquad \text{…(3)}$$

The power series solution to DE (3) will exist and be unique provided that *p(x)* and *q(x)* are analytic in the domain of interest (interval).

Thus, if *p(x)* and *q(x)* have power series representations, then (3) has power series solutions.

5. Power Series Solutions of ODE

Idea of the Power Series Method

For a given equation

$$y''+p(x)y'+q(x)y=0 \qquad \text{...(3)}$$

where $p(x)$ and $q(x)$ are polynomials. We assume a solution in the form of a power series with unknown coefficients,

$$y=\sum_{m=0}^{\infty}a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{...(4)}$$

5. Power Series Solutions to ODE

By term-wise differentiation we obtain y' and y''

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... \qquad (4)$$

$$so, \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + ... \qquad (4a)$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 .... \qquad (4b)$$

5. Power Series Solutions to ODE

Step of the solution

- Substitute equations (4), (4.a) and/or (4.b) into original DE.
- Collect like powers of x.
- Equate the coefficients of each occurring power of x to zero.
- Hence the equation can expressed in power series form.

Example 3:

Solve $y' - y = 0$.

Solution $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \tag{3}$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots. \tag{3a}$$

Insert these into the given differential equation

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + \cdots) = 0.$$

Then we collect like powers of x, finding

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0.$$

Equating the coefficient of each power of x to zero,

$$a_1 - a_0 = 0,\ 2a_2 - a_1 = 0,\ 3a_3 - a_2 = 0, \cdots$$

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2!},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots.$$

Example 3 cont'd…

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2!},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots.$$

With these coefficients, $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ becomes

$$\begin{aligned} y &= a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots \\ &= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \quad \# \\ &= a_0e^x. \end{aligned}$$

⬅ Power series

We can conform the result by using the basic technique: separation of variables → integration.

Example 4:

Solve $y'' + y = 0$.

Solution $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad (3)$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \qquad (3a)$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdot \qquad (3b)$$

By inserting these into the equation we obtain

$$(2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 0.$$

Collecting like powers of x

$$(2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \cdots = 0.$$

Equating the coefficient of each power of x to zero

$$2a_2 + a_0 = 0 \qquad \text{coefficient of } x^0$$

$$3 \cdot 2a_3 + a_1 = 0 \qquad \text{coefficient of } x^1$$

$$4 \cdot 3a_4 + a_2 = 0 \qquad \text{coefficient of } x^2.$$

Solving these equations,

$$a_2 = -\frac{a_0}{2!}, \; a_3 = -\frac{a_1}{3!}, \; a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \cdots$$

Example 4 cont'd…

$$a_2 = -\frac{a_0}{2!},\ a_3 = -\frac{a_1}{3!},\ a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \cdots$$

a_0 and a_1 are arbitrary.

With these coefficients, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ becomes

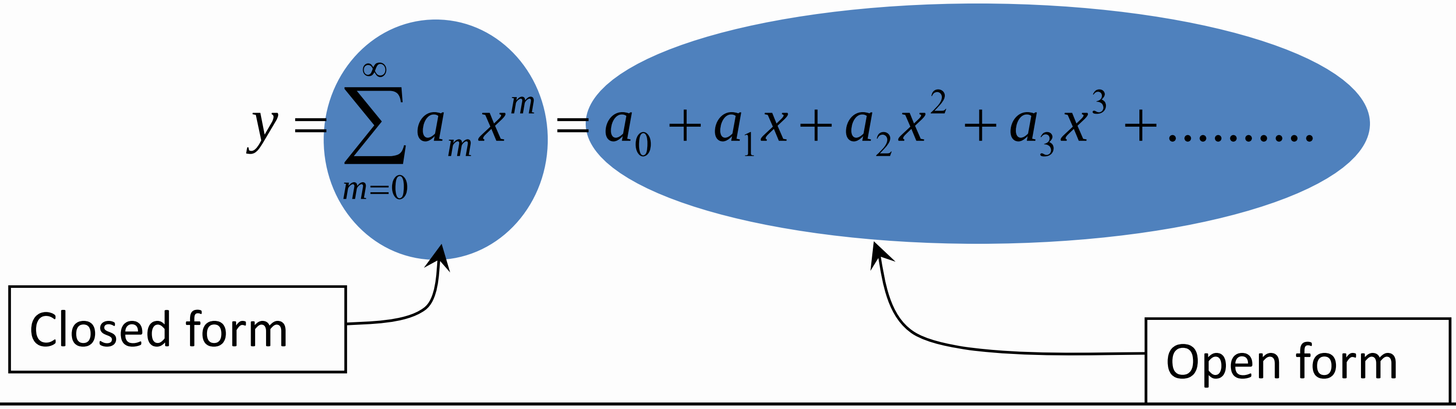
$$y = a_0 + ax_1 - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$$

$$y = a_0\underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right)}_{} + a_1\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)}_{} \quad \#$$

$$y = a_0.\cos(x) + a_1.\sin(x)$$

Note:

1. It is not always necessary to get the power series solution in the form of familiar functions! (trigonometric, exponential, etc).
2. In example 1 and 2, the series solutions are done in open form (the series are expanded). Later we will see how the solutions are done in closed form.



$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$$

Closed form

Open form

Exercise 2:

Solve: $y' + 2xy = 0$

By using power series method!

6. Operations on Power series

(1) **Termwise Differentiation**

If $y = \sum_{m=0}^{\infty} a_m x^m$ is convergent, then

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

are also convergent.

(2) **Termwise Addition**

If $\sum_{m=0}^{\infty} a_m x^m$ and $\sum_{m=0}^{\infty} b_m x^m$ are convergent in the same domain x, then the sum $\sum_{m=0}^{\infty} (a_m + b_m) x^m$ also converges in that domain.

(3) **Termwise Multiplication**

Two power series may be multiplied term by term.

If $\sum_{m=0}^{\infty} a_m x^m$ and $\sum_{m=0}^{\infty} b_m x^m$ are convergent.

Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of x is also convergent.

$$\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \ldots + a_m b_0) x^m$$

(4) **Shifting Summation Indices**

This is technically best explained by example (we will see in the last example).

Examples 5

Solve by power series: $y'' - 2xy = 0.$

Solution:

$x = 0$ is an ordinary point of the equation. From the above Theorem, we can find the solutions of the power series form:

$$y = \sum_{m=0}^{\infty} c_m x^m .$$

$$\Rightarrow y' = \sum_{m=1}^{\infty} m c_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2}$$

Therefore, $$y'' - 2xy = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} - \sum_{m=0}^{\infty} 2 c_m x^{m+1} = 0$$

Examples 5 cont'd…

$$y''-2xy=\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2}-\sum_{m=0}^{\infty} 2c_m x^{m+1}=0$$

Now we will need to shift the first series down by 2 and the second series up by 1 to get both of the series in terms of x^k. Let $k = m - 2$ in the first series and $k = m + 1$ in the second, we have

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}2c_{k-1}x^k=0$$

Next we need to get the two series starting at the same value of index *k*. The only way to do that for this problem is to strip out the *k=0* term.

$$(2)(1)c_2(1)+\sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}2c_{k-1}x^k=0$$

$$2c_2+\sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2}-2c_{k-1}\right]x^k=0$$

Examples 5 cont'd…

We must then have

$2c_2 = 0 \quad \rightarrow c_2=0$ for $k=0$

$(k + 2)(k + 1)c_{k+2} - 2c_{k-1} = 0$ for $k = 1, 2, 3, \ldots$

and

$$c_{k+2} = \frac{2c_{k-1}}{(k+2)(k+1)}, \qquad \text{for } k = 1, 2, 3, \ldots$$

This leads to recurrence relation:

$$c_3 = \frac{2c_0}{3\cdot 2}$$

$$c_4 = \frac{2c_1}{4\cdot 3}$$

$$c_5 = \frac{2c_2}{5\cdot 4} = 0$$

$$c_6 = \frac{2}{6\cdot 5}c_3 = \frac{2^2}{6\cdot 5\cdot 3\cdot 2}c_0$$

$$c_7 = \frac{2}{7\cdot 6}c_4 = \frac{2^2}{7\cdot 6\cdot 4\cdot 3}c_1$$

$$c_8 = \frac{2}{8\cdot 7}c_5 = 0$$

$$c_9 = \frac{2}{9\cdot 8}c_6 = \frac{2^3}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0$$

$$c_{10} = \frac{2}{10\cdot 9}c_3 = \frac{2^3}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}c_1$$

$$\vdots$$

Examples 5 cont'd…

The solution is now:

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x + 0 + \frac{2}{3\cdot 2} c_0 x^3 + \frac{2}{4\cdot 3} c_1 x^4 + 0 + \frac{2^2}{6\cdot 5\cdot 3\cdot 2} c_0 x^6 + \frac{2^2}{7\cdot 6\cdot 4\cdot 3} c_1 x^7$$

$$= c_0 \left[1 + \frac{2}{3\cdot 2} x^3 + \frac{2^2}{6\cdot 5\cdot 3\cdot 2} x^6 + \cdots \right] + c_1 \left[x + \frac{2}{4\cdot 3} x^4 + \frac{2^2}{7\cdot 6\cdot 4\cdot 3} x^7 + \cdots \right]$$

$$y = c_0 y_1(x) + c_1 y_2(x)$$

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Examples 6

We will repeat Example 4, but now the power series will be expressed in Closed form.

Solve it by power series: *y'' +y = 0.*

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow y' = \sum_{m=0}^{\infty} m c_m x^{m-1} = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Substitute them into the equation, therefore:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

Examples 6 cont'd

$$\underbrace{\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}}_{k=m-2} + \underbrace{\sum_{m=0}^{\infty} a_m x^m}_{k=m} = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + a_k] x^k = 0$$

$$(k+2)(k+1)a_{k+2} + a_k = 0$$

For k=0,1,2,……

Since the problem is 2nd order DE, we will let a_0 and a_1 be unknowns (arbitrary), thus:

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$

Examples 6 cont'd

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$

For k=0,1,2,……

$$a_2 = -\frac{a_0}{2!},\ a_3 = -\frac{a_1}{3!},\ a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \cdots$$

With these coefficients, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ becomes

$$y = a_0 + ax_1 - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

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$$y = a_0.cos(x) + a_1.sin(x)$$

Exercise 3:

Solve this ODE using power series: $y'' + xy' + y = 0$

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Thank you
To be continue …